Topology Vol. 32, No. 1, pp. 145–156, 1993.
Printed in Great Britain

0040-9383/93 $6.00 + 0.00

LINK COMPOSITIONS AND THE TOPOLOGICAL SLICE PROBLEM

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(*Received* 22 *March* 1991; *in revised form* 9 *October* 1991)

THE four-dimensional topological surgery "theorem" is still an open question for most fundamental groups. (The class of groups for which the theorem is known coincides with the elementary amenable groups [3], [5].) The general case, is equivalent (Ch. 12, [9]) to the existence of free,‡ flat slices for certain class of links which we have called atomic [2]. There is some choice about the set of atomic links but they are all manifestly iterated compositions. In fact, part of the motivation for searching for new atomic families is the possibility that they will meet a known sufficient condition for the existence of free, flat, slices.

Consider the following list of increasing conditions on smooth links in S^3, the last of which is introduced in this paper:

0. No condition: any link.

$\Uparrow$

1. All linking numbers are trivial.

$\Uparrow$

2. Homotopically trivial: there exists a homotopy from the given link to the unlink so that at no time in the homotopy do *distinct* components cross.

$\Uparrow$

3. Boundary link (∂-link): the link L bounds a Seifert surface Σ with $\pi_0(L) \to \pi_0(\Sigma)$ an isomorphism.

$\Uparrow$

4. Good boundary link. In the definition of boundary link the choice of Σ determines a homomorphism $\pi_1(S^3 \backslash L) \xrightarrow{\phi}$ (free group on meridians). Earlier [6] we defined L to be a good-boundary-link iff (4^-) there exists Σ without $\ker(\phi)$ a perfect group. Here, let us take the stronger definition that there exists a Σ with a standard Seifert form

	$\alpha_1, \ldots, \alpha_g,$	$\beta_1, \ldots, \beta_g$
$\alpha_1, \ldots, \alpha_g,$	0	X
$\beta_1, \ldots, \beta_g$	$I - X$	0

† Supported in part by NSF grant DMS 89-01412.

‡ There exists meridian loops for $(S^3 \backslash link)$ freely generating $\pi_1(D^4 \backslash slices)$.

where $X_{ij} = 0$ if $i \neq j$ and $X_{ii} = 0$ or 1, $1 \leq i, j \leq g$. It may be a small but interesting open problem whether $(4^-) \Leftrightarrow (4)$, certainly $(4) \Rightarrow (4^-)$.

$\Uparrow$

5. Boundary squared link ($\partial^2 - link$). This is good boundary link for which $(\Sigma, (\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g))$ can be chosen with trivialized Seifert form, as in (4), and further to the 1/2 basis $\alpha_1, \ldots, \alpha_g$ is represented by disjoint simple closed curves $a_1, \ldots, a_g$ with $\bigcup_{i=1}^{g} a_i = A = \Sigma \cap \Sigma' = \partial\Sigma'$ for a second surface Σ' meeting Σ transversely and satisfying $\pi_0(A) \to \pi_0(\Sigma')$ is an isomorphism.

It is known that the 4-dimensional, topological surgery conjecture is equivalent to "all good boundary links are free, flat, slice." Here we prove (Theorem 1.1) all ∂^2-links are free, flat, slice. A special case of this was obtained earlier [6]: the Whitehead double of a boundary link is free, flat, slice. One new application is a class of atomic links which can be "sliced" by flat annuli—rather than disks.

Link composition $J \circ L$, originating in a construction of J. H. C. Whitehead [12], makes sense given a link L in S^3 and a link J_i in a framed solid torus $(S^1 \times D^2)$; for each component L_i of L replace that component with J_i by identifying an untwisted neighborhood of L_i with $S^1 \times D^2$. Atomic links, devolve from the 1-handles of a handle diagram. As a result *all* the links J_i, which we will encounter, have the additional feature that when $S^1 \times D^2$ is included into S^3 in the standard way it becomes the unlink. Also, for convenience, we extend definitions (1), (2), and (3) to links in $S^1 \times D^2$ in the obvious manner: all components are required to be trivial in $H_1(S^1 \times D^2; Z)$; all homotopies and imbedded surfaces must lie in $S^1 \times D^2$.

The standard atomic links [5], [9] are compositions of form $Wh(Bing^n(Hopf))$, or $(3) \circ (1)^n \circ (0)$. They satisfy (4) but appear not to satisfy (5). Using what exists of the non-simply-connected theory it is possible (Theorem 2.1) to produce a new atomic family whose links are of the form $(2)^m \circ (3) \circ (1)^n \circ (0)$. These links seem tantalizingly close to satisfying condition (5) which roughly speaking is $(3)^2$.

Another way of expressing the underlying idea is to define 1/2-Casson handles: infinite towers containing *only* the Whitney disks and not the accessory disks [9]:

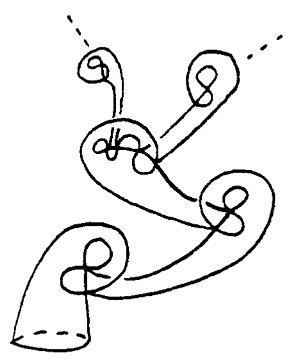


Fig. 1.

Theorem 2.2 states that any good boundary link bounds disjointly imbedded 1/2-Casson handles in $S^3 \times [0, \infty)$. Unlike Casson handles (CH), we know that a 1/2-CH *cannot* contain an imbedded disk with boundary the attaching circle. On the other hand, an infinite surgery procedure known as *inflation* [1] shows that any link bounding disjointly imbedded 1/2-CH's in $S^3 \times [0, \infty)$ is flat slice in a simply connected Z-homology $S^3 \times [0, \infty)$ (the end changes) and there is no known example where such a link does not bound flat slices in D^4.

I thank the referee for several helpful comments.

§1. A LINK IDENTITY WITH APPLICATIONS TO THE LINK SLICE PROBLEM

First consider two curves a and b in S^3 meeting in one (maximally transverse) crossing. Assigning integral framings to a and b determines a two-dimensional thickening (up to isotopy) which is an imbedded punctured torus $T^- \subset S^3$. The punctured torus is a function of this data, $T^-(a,\ \mathit{frame}(a),\ b,\ \mathit{frame}(b))$. Let $\mathscr{S}^n$ denote n-framed surgery on a knot or link, we have:

Warm up identity: $\mathscr{S}^0(\partial T^-(a, n, b, 0) = \mathscr{S}^n(a)$/round 0-surgery. The "round 0-surgery" is constructed by: (1) Forming disjoint copies of b, b^+, and b^- by pushing b (positively and negatively) normal to T^-. (2) Delete the interior of closed regular neighborhoods $\mathscr{N}^+(b^+)$

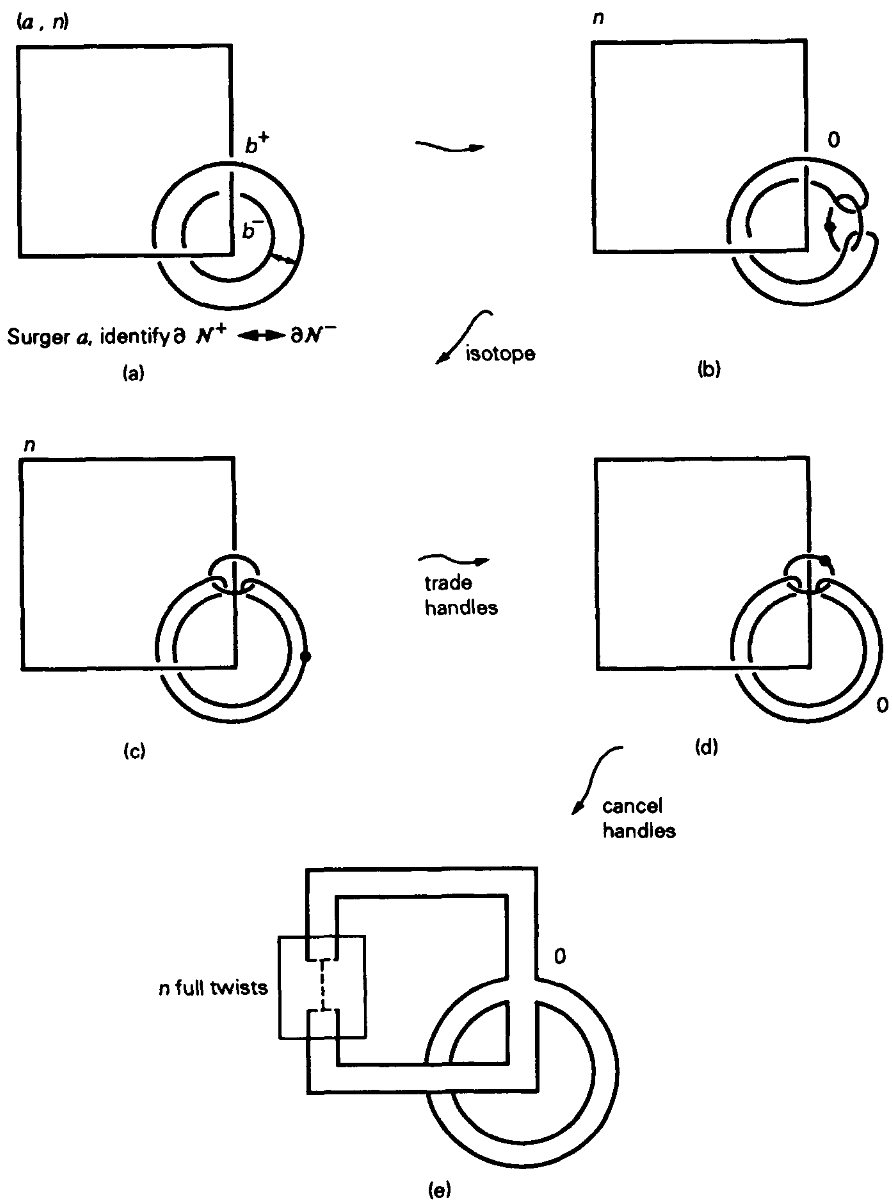


Fig. 2.

and $\mathcal{N}^-(b^-)$. And (3) Glue $\partial\mathcal{N}^+$ to $\partial\mathcal{N}^-$ so as to identify meridian with meridian and longitude with longitude.

Proof: The two geometric shapes above stand for arbitrary simple closed curves in S^3. The preceding diagrams represent equivalences of 3-manifolds.

The general (high genus, several component) case described below requires more notation but its proof is parallel to the "warm up" and can safely be left as an exercise. Let $a_1^k, \ldots, a_{j_k}^k, b_1^k, \ldots, b_{j_k}^k$, $1 \leq k \leq K$ be smooth simple closed curves in S^3, disjoint except that $a_i^k \cap b_i^k$ in one (maximally transverse) point x_i^k. Assign framing frame $(a_i^k) = n_i^k$ and frame $(b_i^k) = 0$. Let $c_1^k, \ldots, c_{j_k}^k$ be framed arcs c_i^k joining b_i^k to a base disk Δ^k so that the first frame vector to b_i^k and the inward pointing normal to $\partial\Delta^k$ are tangent to c_i^k at its beginning and end point respectively. Thicken the connected pieces according to framings n_i^k along a_i^k, framing 0 along b_i^k and the specified framing along c_i^k. The result is surface $\Sigma = \amalg_k \Sigma^k$. We have the following.

Identity. $\mathscr{S}^0(\partial\Sigma) = \mathscr{S}^{n_i^k}(a_1^1, a_2^1, \ldots, a_{j_K}^K)$/generalized round 0-surgeries.

The "generalized round 0-surgery" may be described as follows. Let $b \cup c$ be the 1-complex consisting of circles and arcs which, apart from sub- and superscripts, are denoted by b or c. Let $\mathcal{N}^+(\mathcal{N}^-)$ be closed regular neighborhoods of $b \cup c$ displaced positively (negatively) from Σ. Delete interior $\mathcal{N}^+$ and interior $\mathcal{N}^-$ and identify $\partial\mathcal{N}^+$ with $\partial\mathcal{N}^-$ by the diffeomorphism induced by displacement normal to Σ. □

Writing $\mathcal{N}^+(\mathcal{N}^-)$ as a union of components we have $\mathcal{N}^+ = \mathcal{N}_1^+ \cup \ldots \cup \mathcal{N}_S^+$ and $\mathcal{N} \equiv \mathcal{N}_1^- \cup \ldots \cup \mathcal{N}_S^-$.

Now consider the case L is a ∂^2-link. From the definition and the identity and $L = \partial\Sigma$ where $\{a_1^1, \ldots, a_{j_K}^K\} = \{a's\}$ is itself a boundary link as in (5). There is an integral homology equivalence f [6] which when covered by a bundle map becomes a degree $= 1$, normal map. There is a well defined "Arf invariant" $\in H_2(\mathscr{S}^0(\{a's\}); Z_2)$ obstructing the extension of f to a degree $= 1$, normal map g as indicated in the diagram below.

$$\begin{array}{ccc} \mathscr{S}^0(\{a's\}) & \xrightarrow{f} & \mathop{\#}\limits_{i=1}^{K} S^1 \times S^2 = X \\ \partial \big\downarrow & & \partial \big\downarrow \\ N & \xrightarrow[g]{} & \mathop{\natural}\limits_{i=1}^{K} S^1 \times D^3 = Y. \end{array}$$

If g exists it may be normally borded (rel ∂) to an isomorphism on π_1. For the present, we assume the obstruction vanishes and that (N, g) exist and that g is an isomorphism on π_1. Eventually we will see that this obstruction is irrelevant. It does not actually obstruct the construction of a slice complement for $L = \partial\Sigma$ but the existence of a subsidiary structure (a certain splitting) on that manifold.

Let $F(S)$ be the free group on the symbols $1, \ldots, S$. Define $\tilde{M} = N \times F(S)/\mathcal{N}_s^+ \times g \equiv \mathcal{N}_s^- \times sg$. Note that $F(S)$ acts freely on $\tilde{M}$ and call the quotient M. Observe that ∂M obtained from ∂N by generalized round 0-surgeries on along $(b \cup c)^\pm$ (using the notation of the identity for the standard curves on Σ).

From the identity we have $\partial M = \mathscr{S}^0(L)$. Furthermore, if D_s^+ and D_s^-, $1 \leq s \leq S$, are disjoint 3-cells in the boundary of the 4-ball then the obvious equivariant map from $\tilde{M}$ to

$B^4 \times F(S)/D_s^+ \times g \equiv D_s^- \times sg$ descends to a degree $= 1$ normal map

$$(M, \partial M) \xrightarrow{h} \left(\mathop{\natural}\limits_{i=1}^{S} S^1 \times D^3, \partial \right)$$

Since $g_\sharp$ is an isomorphism on π_1, a loop in ∂N is null homotopic in N if it is disjoint from the surfaces $\bar{\Sigma} < \partial N$ made by capping off Seifert surfaces Σ'. Each b has one normal push off $b^\flat$ (+ or $-$) which is disjoint from $\bar{\Sigma}$ and another $b^\sharp$ so that $\{b^\sharp\}$ when appropriately based, are free generators for $\pi_1(Y)$. Since each generator $b^\sharp$ dies by becoming a $b^\flat$ in an adjacent fundamental domain, the map h is also an isomorphism on π_1. Furthermore, the kernel group K_2 is simply the integral kernel of g, $H_2(N; Z)$, together with its integral intersection form λ: $H_2(N; Z) \otimes H_2(N; Z) \to Z$, tensored over Z with $Z[F(S)]$. Since $N \cup (B^4 \cup_{\{a's\} \text{ with zero framing}}$ 2-handle's) is a closed spin 4-manifold, it follows from Rochlin's theorem that λ is even nonsingular and has signature divisible by 16. Changing N by connected sum with copies of $\pm$ Kummer surface we may assume $\sigma(\lambda) = 0$ and therefore that λ is a sum of standard planes. It follows that $\lambda \otimes_Z Z[F(S)]$ carries the trivial surgery obstruction. Thus the surgery conjecture, if true, would imply that h is normally cobordant to a homotopy equivalence (rel ∂): $(M^*, \partial M^*) \xrightarrow{h^*} \left(\natural_{i=1}^{s} S^1 \times D^3, \partial \right)$. This would be the desired (closed) slice complement for L, showing L has free, flat, slices.

In this case, the unresolved status of surgery conjecture is no obstacle. In [7] we show that if a standard surgery kernel is represented by maps of spheres into a codimension = 0 submanifold (interior N) whose inclusion into the domain of the surgery problem (M) is trivial on π_1. then surgery can, in fact, be completed.

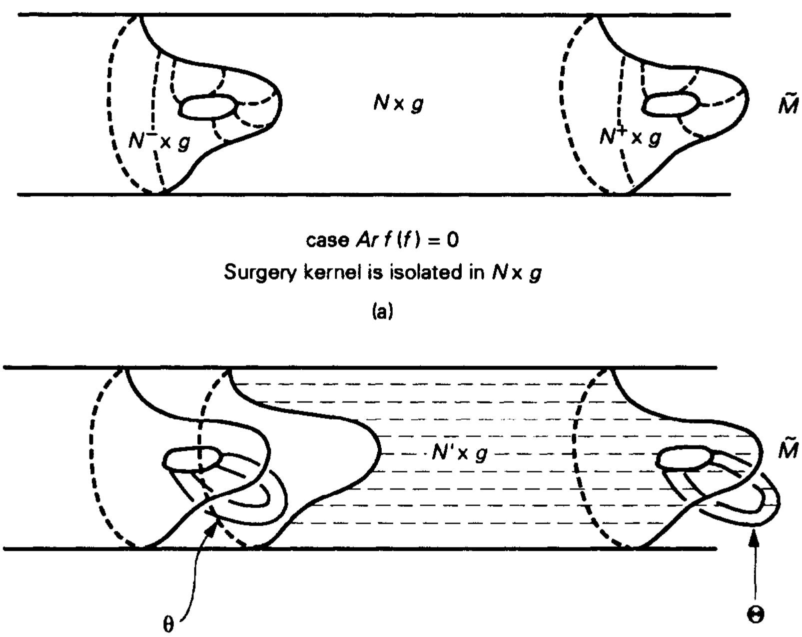
Finally we describe the construction of M, with a π_1-trivial surgery kernel, in the case that the Arf invariant in $H_2(\mathscr{S}^0(\{a's\}); Z_2)$ is non-trivial.

For brevity write $\mathscr{S}^0(\{a's\}) = A$. Form $\tilde{V} = (A \times I) \times F(S)/\mathscr{N}_s^+ \times 1 \times g \equiv \mathscr{N}_s^- \times 1 \times sg$. Clearly $F(S)$ acts freely on $\tilde{V}$ with fundamental domain $A \times I \times g$ and quotient V. We now fix g and let s vary. By assumption (5) on the Seifert form, the collection of curves $b^\flat \times 1$ bounds imbedded surfaces $\{\Theta'\}$ in $A \times 1$ whose interiors are disjoint from $b^\flat \times 1$, and $b^\sharp \times 1$. Push the interiors of $\{\Theta'\} \times g$ into the interior of $A \times I \times g$ to obtain Θ, a (disconnected) imbedded surface consisting of null homologies for $b_s^\sharp \times 1 \times g = b_s^\flat \times 1 \times sg$ in $A \times I \times sg$. We define θ to be the intersection of the $F(S)$ covering translations of Θ with $A \times I \times g$; and similarly θ' to be the intersection of translates of Θ'. The surface θ consists of null homologies for $b_s^\sharp \times 1 \times s^{-1}g = b_s^\flat \times 1 \times g$ in $A \times I \times g$. Furthermore $\theta' \cap b^\sharp = \emptyset$. Let $E \subset V$ be the 3-manifold which results by ambient "surgery" on A along $\Theta \times D^2$ and projecting to V. To ensure that E is imbedded, exploit $\Theta' \cap b^\sharp = \emptyset$ to isotope a portion of A containing θ' along the I coordinate as in Fig. 3(b). Consider the effect of attaching $\Theta \times D^2$ to the domain and range of f. The result is a Z-homology equivalence $\bar{f}$: $E \to \bar{X} = \sharp S^1 \times S^2$'s where the number of factors is 2 (genus Θ). But $Arf(\bar{f}) = 0 \in H_2(\bar{X}; Z_2)$ since the second homology of E is generated by tori = (circle in Θ) $\times$ (normal circle) which are normally bounded by 3-manifolds = (circle in Θ) $\times$ (normal disk). As before, this enables E to be "filled-in" with a normal map $(N', \partial N' = E) \xrightarrow{g'} (\natural\Theta \times D^2 = \natural S^1 \times D^3$'s, $\partial = \bar{X})$. Again the fundamental group can be adjusted and the signature of N' can be reduced to zero, so that $ker(g')$ is a standard sum of planes represented by immersed spheres.

Form M' by replacing a complementary region of E in V (the one shaded in Fig. 3(b) with N'. As before there is a normal map h': $M' \to Y$ and $ker_2(h')$ is a standard surgery kernel for M' satisfying the π_1-null condition.

Completing surgery, we obtain M'^*. Either M^* or M'^* will serve as the closed slice complement. This proves:

THEOREM 1.1. *Any ∂^2-link admits free, topologically flat 2-disk slices in B^4.* □



case $Arf(f) = 0$
Surgery kernel is isolated in $N \times g$
(a)

Case $Arf(f) \neq 0$, surgery kernel is isolated in $N' \times g$
(b)

Fig. 3.

We conclude this section with an application suggested by the theory of capped gropes ([5] and [9]). The tip of a capped grope has the form:

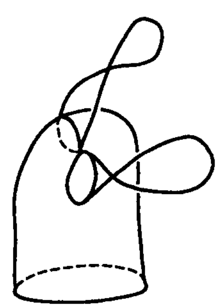


Fig. 4.

That is kinky handles are attached to a symplectic basis of the final grope stage. Standard methods of stabilization and construction of dual spheres (Ch. 1–3, [9]) permit additional (disjoining, framed) null homotopies to be added to $(+g, -g)$ pairs of double points on dual caps. Schematically such "extended" capped gropes would terminate as shown in Fig. 5.

This leads to its own link slice problem—also intractable—but if the extra caps happen to be imbedded the associated link slice problem is covered by Theorem 1. In fact the imbedded case also demonstrates that the underlying link slice problem of the capped grope can be "solved" with (flat) *annular* slices connecting pairs of components instead of disks bounding each component individually.

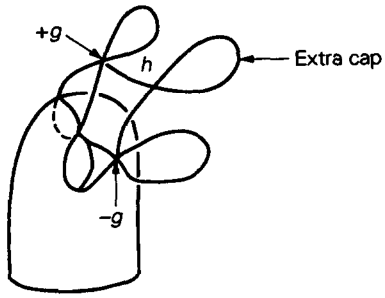


Fig. 5.

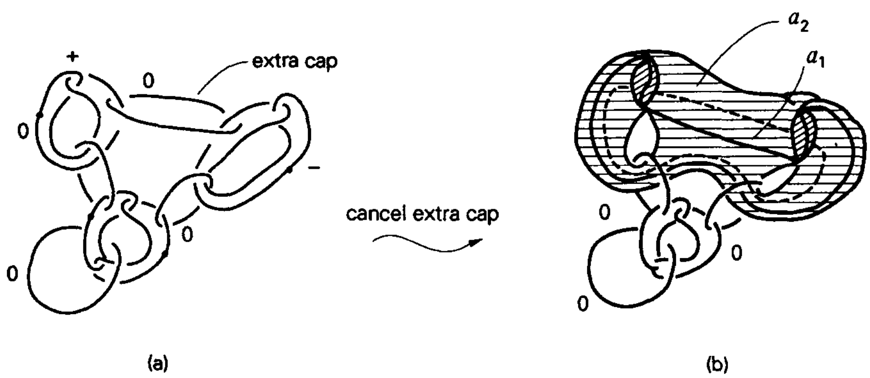


Fig. 6.

Above, we indicate the link calculation near the tip of an extended capped grope in which the extra caps are framed imbeddings.

The upper component bounds the shaded genus two surface Σ. There is a symplectic basis of curves a_1, a_2, b_1, b_2 on Σ. (a_1 and a_2 are drawn in Fig. 6(b).) The curves a_1 and a_2 bound secondary surfaces as in condition (5). As an aid to visualizing this observe the position of $a_1 \cup a_2$ after Morse cancelling the two 1-2 handle pairs. They become two parallel copies of a Whitehead curve linking the initial 2-handle:

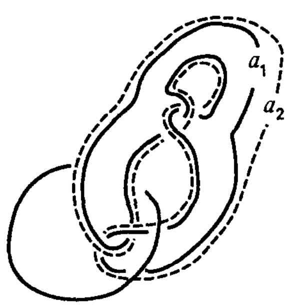


Fig. 7.

This is a boundary link in the solid torus linking the 2-handle curve. The link slice diagram of an "extended capped grope" of sufficient height will be a ∂^2 link provided the extra caps are non-singular. In the singular case, additional "Whitehead components" appear to destroy the ∂^2-structure.

§2. EXTENSIONS OF CAPPED GROPES

Let F be the free group on $x_1, \ldots, x_n$. Let $M = F/\langle [x_i, x_i^g] \rangle$ the quotient of F by the normal subgroup generated by the relations that all generators commute with their conjugates. The following lemma essentially comes from Milnor [11].

LEMMA 2.1. *M is a finitely presented nilpotent group. In particular $n + 1$-fold commutators are trivial in M.*

Proof. If $n = 1$, $M \cong Z$ and two-fold commutators $[x_1, x_2]$ are trivial. Assume the lemma is proved for $n - 1$. Consider the quotient maps θ_i, $i = 1, \cdots, n$ in which x_i is made a relator, the image is isomorphic to M_{n-1} and will be denoted by M_n^i. Since $ker\ \theta_i$ is normally generated by $x_i \in M$, the relations make $ker\ \theta_i$ abelian. But $\cup_i\ ker\ \theta_i$ generates all of M so $C = \cap_i\ ker\, \theta_i$ is central in M. Consider the exact sequence

$$0 \to C \to M_n \xrightarrow{\oplus\theta_i} \oplus_i M_{n-1}^i.$$

Inductively, the group on the right has all its n-fold commutators q trivial. Thus all n-fold commutator in M_n lies in the central subgroup C. Consequently $n + 1$-fold commutators are trivial.

For N a finitely generated nilpotent group the extension

$$[N,N] \to N \to N/[N,N]$$

is a quotient of similar extension for a "free-nilpotent" $\bar{N}$ for which the outside terms are well known [10] to be finitely generated. Thus the outside terms above are finitely generated. Replacing this exact sequence with a fibration of $K(\pi_1 1)$'s we may suppose by induction on nilpotence height that $K([N,N],1)$ and $K(N/[N,N],1)$ have finite 2-skeletons. Thus $K(N, 1)$ also has a finite 2-skeleton and N has a finite presentation. The next lemma is proved in [5], also see [9].

LEMMA 2.2. *Given a capped grope G (of group height ≥ 3) and a homomorphism $\pi_1(G) \xrightarrow{h} N$ to a nilpotent group. There exists a "π_1-negligible reimbedding" of the upper stages so that the new capped grope $G' \subset G$ satisfies $inc_{\#}\ \pi_1(G') \subset ker\ h$.* □

Combining the two lemmas we see that a capped grope (of grope height ≥ 3) can be replaced by $\bar{G} \subset G$, a capped grope with special 2-complexes K attached to the generating loops of the caps. K is a surface $\Sigma \cup$ cylinders: given a standard simplectic basis of loops $a_1, \ldots, a_g, b_1, \ldots, b_g$ and cylinders $c_1, \ldots, c_g$ have $\partial c_i = a_i \cup b_i$, $i = 1, \ldots, g$. K may be singular but piping double points reduce all singularities to double points of Σ.

Since K admits framed immersed dual spheres, framings may be standardized to agree with the next figure (Fig. 9). Also the extension of G can be iterated so that new (singular)

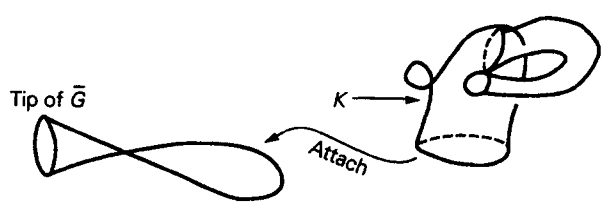


Fig. 8.

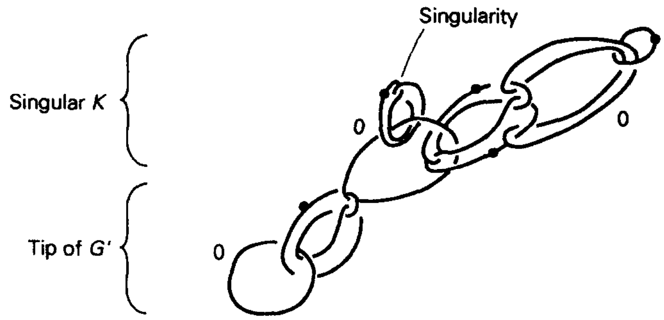


Fig. 9.

K's are attached to loops freely generating π_1 (image K). This process may be carried to any finite stage n resulting in $\bar{G}^n \subset G$.

The handle diagram for a tip of $\bar{G}$ is shown below in the case genus $(\Sigma) = 1$ and Σ has one (+) double point (as in Fig. 8).

Cancelling the upper 2-handle with either of two 1-handles we obtain:

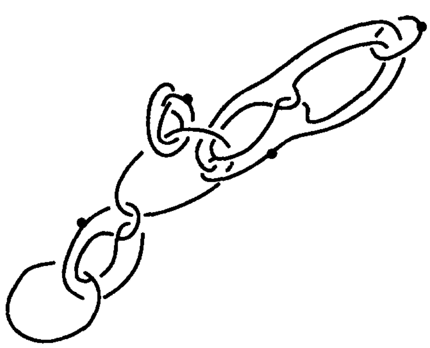


Fig. 10.

Notice that cancelling the remaining Morse pair would change the link diagram by a link composition where a link homotopically trivial in $S^1 \times D^2$ (and trivial in S^3) is inserted around each 1-handle curve of G'. This produces a fragile looking class of atomic links:

THEOREM 2.1. *Given any integers m, n > 0. There exists a class of atomic links, i.e., links for whom the free slice problem is equivalent to the general surgery conjecture, of the form:* $(2)^m \circ (3) \circ (1)^n \circ (0)$. *That is m-homotopically trivial compositions of a Whitehead doubling (i.e., boundary composition) of n Bing doublings (i.e., linking no = 0 composition) of the Hopf link.*

Proof. Start with a capped-grope-$S^2 \times S^2$ of sufficient grope height. Then replace the two gropes G_1 and G_2 with m-fold extended objects $\bar{G}_1^m$ and $\bar{G}_2^m$. Compute the link diagram: the key illustration is Fig. 10. □

Next we define finite and infinite "1/2-Casson towers." In the notation of [9], they are built from singular "Whitney disks" with the "auxiliary disks" omitted. Schematically they are represented in Fig. 1. More precisely they are defined by the class of 4-manifold handle diagrams indicated in Fig. 11. The case of least multiplicity is drawn explicitly.

Here is an important difference between a 1/2-CH and a CH.

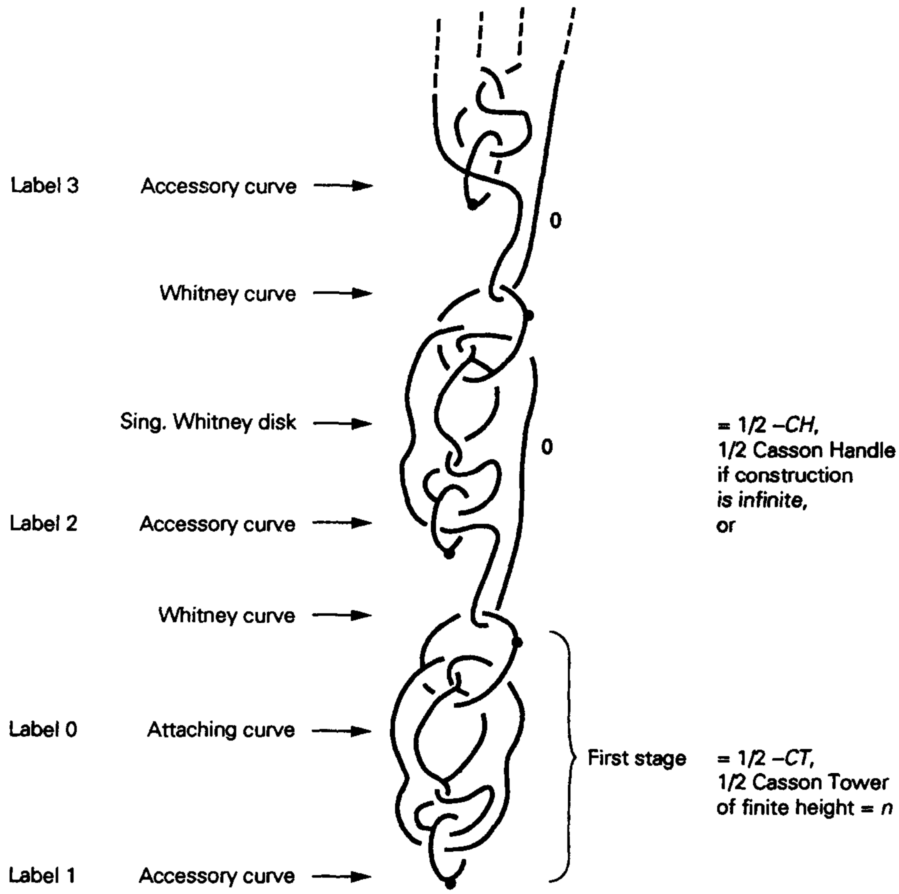


Fig. 11.

Fact 2.1. In no $1/2$ *CH* can the attaching curve bound an imbedded (even non-locally flat) disk.

Proof. By compactness we may work, without loss of generality in a finite $1/2$-CT. Also, there is no loss of generality in considering the lowest multiplicity (unbranched) case (Fig. 11) since 2-handles may be attached to an arbitrary $1/2$-CT to reduce to this case.

Let 0 label the attaching curve and $1, \ldots, n$ label the accessory curves in the diagram (Fig. 11) for the unbranched height $= n$ $1/2 - CT$ and let the $n+1$ label the highest Whitney curve. The other curves labeled "Whitney curve" and "singular Whitney disk" may be cancelled so that Fig. 11 becomes an $n+2$-component link L^n with labels $0, 1, \ldots, n, n+1$. It is sufficient for us to show that any $\bar{\mu}$ invariant of L^n is non-zero. Cochran's imbedded surface method [4], as explained to me by Kevin Walker, may be applied to compute:

$$\bar{\mu}_{L^1}(0,1,0,2) = 2$$

$$\bar{\mu}_{L^2}(0,1,0,2,0,1,0,3) = 2^2$$

$$\bar{\mu}_{L^3}(0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,4) = 2^3$$

$$\vdots$$

$$\bar{\mu}_{L^n}(0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,4,0,1,0,2,0,\ldots,n+1) = 2^n.$$

The possibility of indeterminacy does not bother us since this only arises if some shorter $\bar{\mu}$ is non-zero. This computation completes the proof. □

On the other hand we have the following counterpoint:

Fact 2.2. Any link in S^3 which bounds disjoint properly imbedded 1/2-*CH*'s in $S^3 \times [0, \infty)$ is smoothly slice in a smooth simply connected 4-manifold M with $\partial M = S^3$ and $inc_* \, H_*(S^3; Z) \to H_*(M; Z)$ an isomorphism.

Proof. The idea goes back to Cannon's "inflation" [1]. To construct M perform an infinite sequence of (0-framed) surgeries on all the Whitney circles of the 1/2 *CH*'s which are above the bottom stage. The bottom stages now are regularly homotopic to slice disks. Consider the spheres made from the two possible Whitney disks for each Whitney circle, the original singular Whitney disk contained in 1/2-*CH* and the new one introduced by surgery on the Whitney circle. These spheres are regularly homotopic (use the new Whitney disks) to disjoint imbeddings (with dual spheres) and surgery on these homologically cancels the surgeries on the Whitney circles (and preserves $\pi_1 = 0$) to produce M. Because the spine of the 1/2-*CH*'s may not possess disjoint dual spheres in $S^3 \times [0, \infty)$, the end of M may develop a bad fundamental group at infinity and fail to be homeomorphic to $S^3 \times [0, \infty)$. Since M is non-compact, there is no obstruction to extending a smoothing near the link slice across all of M. □

There is no known example of a (smooth) link in S^3 which bounds disjoint 1/2-*CH*'s in $S^3 \times [0, \infty)$ but which is not (flat) slice in $S^3 \times [0, \infty)$. Furthermore, the hypothesis of Fact 2.2 can often be achieved.

THEOREM 2.2. *Let $L \subset S^3$ be a good boundary link (condition 4) or even one satisfying the weaker algebraic condition 4^-. Then L bounds disjoint properly imbedded 1/2-CH's in $S^3 \times [0, \infty)$.*

Proof. According to [6] hypothesis (4^-) allows the construction of an unobstructed smooth surgery problem g (inducing an isomorphism on π_1)

$$(M^4, \mathcal{S}^0(L)) \xrightarrow{g} (\natural S^1 \times D^3\text{'s}, \sharp S^1 \times S^2\text{'s})$$

The kernel K_2 is represented by disjoint capped-grope-$S^2 \vee S^2$'s $= W$. Attach 0-framed 2-handles to the small linking circles to L to form $M^+ = M^4 \cup$ 2-handles.

By Lemma's 2.1 and 2.2 we may choose W so that if self-finger moves are performed on the co-cores, c, of the 2-handles (call the result c') to introduce sufficiently many relations of the form $[x_i, x_i^g]$ then a second layer of caps may be added to W (call the result W^+) disjoint from c'. Let c_1 be the Whitney disks for c'. We now work inside W^+ by performing self finger moves to the subdisks $c_1 \cap W^+$. Since $\pi_1 W$ is contained in the free group generated by meridians to these subdisks, Lemmas 2.1 and 2.2 apply. After self finger moves on c_1 (to c'_1) there will be a reimbedding $W_1 \subset W$ which may be capped off in W^+ to form a reimbedding $W_1^+ \subset W^+$ so that $W_1^+ \cap (c' \cup c'_1) = \emptyset$, W_1^+ is π_1-negligible in W^+, and $\pi_1(W_1^+) \to \pi_1(W^+)$ is zero. (The last two conclusion are the chief consequence of having two layers of caps: one exploits the abundant dual spheres below the top layer of caps to achieve both these conditions). Let c_n be the Whitney disks for c'_{n-1} and similarly construct a reimbedding W_n^+ disjoint from $(c' \cup \ldots \cup c'_n)$ with W_n^+ π_1-negligible in W_{n-1}^+ and $\pi_1(W_n^+) \to \pi_1(W_{n-1}^+)$ trivial.

If each W_i^+ is made connected by adding a minimal collection of 1-handles, preserving the inclusions $W_{i+1}^+ \subset W_i^+$, then $M^+ \setminus (\cap_{i=1}^{\infty} W_i^+ \cup \text{1-handles})$ will be proper homotopy equivalent and therefore homeomorphic [8] to $S^3 \times [0, \infty)$. The union $c' \cup c'_1 \cup \ldots$ is the spine of the disjoint collection of imbedded $1/2$-CH's which was to be constructed. □

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